Singular structure of the QED effective action

Is there exist a free electron?

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Outline

Are quantum field models analytic in coupling constant?

Are vacuum expectations or effective action analytic in coupling constant? Expanding amplitude of any process in perturbation series over coupling constant we implicitly suppose analyticity of this amplitude in this coupling constant. But it is well known that perturbation series in quantum field models are asymptotic ones. We want to investigate perturbation series in QED from the following viewpoint: is there any singularity in this series at $e = 0$, where e is electric charge (coupling constant). I.e., can we "switch off" the electromagnetic interaction in Quantum Electrodynamics?

General definitions

We will use the following notations: the effective action $W[\eta, \bar{\eta}, J_\mu]$ which is generator of all weakly connected Green functions, is connected with the generator of all Green functions $Z[\eta, \bar{\eta}, J_\mu]$ (partition function) as follows:

$$Z[\eta, \bar{\eta}, J_\mu] = \exp\left(iW[\eta, \bar{\eta}, J_\mu]\right).$$

Here $J_\mu, \eta, \bar{\eta}$ are external classical sources. Consequently, the vacuum expectations (in the presence of external sources) for quantum fields are

$$\psi_i = \frac{\delta W}{\delta \bar{\eta}^i}, \qquad \bar{\psi}_i = -\frac{\delta W}{\delta \eta^i} \qquad \text{and} \quad A_i^\mu = \frac{\delta W}{\delta J_\mu^i}.$$

Equations

Applying the DeWitt operator

$$\Lambda = 1 - \frac{i\hbar}{2}\frac{\delta^2 W}{\delta J^i_\mu \delta J^j_\nu}\frac{\delta^2}{\delta A^\mu_i \delta A^\nu_j} + i\hbar\frac{\delta^2 W}{\delta J^i_\mu \delta \eta^j}\frac{\delta^2}{\delta A^\mu_i \delta \bar{\psi}_j} - i\hbar\frac{\delta^2 W}{\delta J^i_\mu \delta \bar{\eta}^j}\frac{\delta^2}{\delta A^\mu_i \delta \psi_j} + i\hbar\frac{\delta^2 W}{\delta \eta^i \delta \bar{\eta}^j}\frac{\delta^2}{\delta \psi_j \delta \overline{\psi}_i} + \cdots$$

to the

$$\Lambda\frac{\delta S}{\delta A^i_\mu} = -J^\mu_i, \qquad \Lambda\frac{\delta S}{\delta \psi^i} = \bar{\eta}_i, \qquad \Lambda\frac{\delta S}{\delta \bar{\psi}^i} = -\eta_i$$

where S is classical action for QED

we get following functional equations for the QED effective action W (here all the fields are vacuum expectations of quantum fields):

$$-e\bar{\psi}\gamma^{\mu}\psi + D^{-1\mu\nu}A_{\nu} + ie\hbar\,\mathrm{Tr}\left(\gamma^{\mu}\frac{\delta^2 W}{\delta\overline{\eta}\delta\eta}\right) = -J^{\mu};$$

$$\left[\bar{\psi}(x)\left(i\overleftarrow{\widehat{\partial}} + e\widehat{A} + m\right)\right]_{\alpha} + ie\hbar\left(\frac{\delta^2 W}{\delta J^i_{\mu}\delta\eta^i}\gamma_{\mu}\right)_{\alpha} = \overline{\eta}_{\alpha};$$

$$\left[\left(i\widehat{\partial} - e\widehat{A} - m\right)\psi\right]_{\alpha} + ie\hbar\left(\gamma_{\mu}\frac{\delta^2 W}{\delta J^i_{\mu}\delta\overline{\eta}^i}\right)_{\alpha} = -\eta_{\alpha}.$$

B.Fayzullaev, M.Musakhanov. Annals of Physics, (1995).

Due to

$$\frac{\delta^2 W}{\delta J^i_\mu \delta \eta^j} = -\frac{\delta \bar{\psi}^j}{\delta J^i_\mu} = \frac{\delta A^\mu_i}{\delta \eta^j}; \quad \frac{\delta^2 W}{\delta J^i_\mu \delta \bar{\eta}^j} = \frac{\delta \psi^j}{\delta J^i_\mu} = \frac{\delta A^\mu_i}{\delta \bar{\eta}^j}; \quad \frac{\delta^2 W}{\delta \overline{\eta}^\alpha \delta \eta^\beta} = -\frac{\delta \overline{\psi}^\beta}{\delta \overline{\eta}^\alpha} =$$

the equations may be bringed to the more convenient for further consideration form:

$$ie\hbar\,\mathrm{Tr}\left(\frac{\delta \bar{\psi}^i}{\delta \bar{\eta}^i}\gamma_\mu\right) = ie\hbar\,\mathrm{Tr}\left(\frac{\delta \psi^i}{\delta \eta^i}\gamma_\mu\right) = J^i_\mu - e\bar{\psi}\gamma_\mu\psi + D^{-1}_{\mu\nu}A^\nu;$$

$$-ie\hbar\frac{\delta \bar{\psi}^i}{\delta J^i_\mu}\gamma_\mu = ie\hbar\frac{\delta \widehat{A}}{\delta \eta} = \bar{\eta}^i_\alpha - \left[\bar{\psi}(x)\left(i\overleftarrow{\widehat{\partial}} + e\widehat{A} + m\right)\right]_\alpha;$$

$$ie\hbar\gamma_\mu\frac{\delta \psi^i}{\delta J^i_\mu} = ie\hbar\frac{\delta \widehat{A}}{\delta \bar{\eta}} = -\eta^i_\alpha - \left[\left(i\widehat{\partial} - e\widehat{A} - m\right)\psi\right]_\alpha.$$

I.e., they are equations for vacuum expectations of quantum fields.

Usual way to solve these equations is perturbative expansion over small coupling constant e. I.e., at first step we put coupling constant $e = 0$, thereby turning these equations into equations for free particles which can be solved easily. Further, interactions between free particles are taken into account iteratively over small parameter e (really, $e^2/4\pi$) getting perturbative power series:

$$W = W_0 + eW_1 + e^2 W_2 + \cdots . \tag{1}$$

This is usual way. But there is one circumstance must be taken into account. It is obvious that in front of each derivative term there is coupling constant e, this means if we put $e = 0$ then our (variational) differential equations transform to algebraic ones. In the following section we show that in this case a full solution to equations of such type (with small parameter) should contains not only regular but singular (in e) part too.

A simple example

The formulation of problem under consideration may be explained in the following simple example: find solution to equation

$$\mu\frac{dx(t)}{dt} = a(t)x(t) + b(t), \qquad x(0) = x_0, \qquad 0 \leq t < \infty$$

in the form of perturbative expansion over small μ. It is very easy to find exact solution to this equation obeying given boundary condition:

$$x(t) = x_0 \exp\left(\frac{1}{\mu}\int_0^t a(s)ds\right) + \frac{1}{\mu}\int_0^t b(s)\exp\left(-\frac{1}{\mu}\int_t^s a(z)dz\right) ds.$$

A problem arise

It is obvious that $\mu = 0$ is essential singular point for solution to our eqs., and, consequently, regular perturbative expansion for small μ can not exist. The reason for such situation can be seen from the equation itself: if we put $\mu = 0$ in this equation then it fails to be differential equation and becomes to be algebraic one

$$a(t)\tilde{x}(t) + b(t) = 0. \tag{2}$$

But solution to this (algebraic) equation $\tilde{x}(t) = -b(t)/a(t)$ in general can not obey given boundary condition: $a(0)/b(0) \neq x_0$. It happens loss of boundary condition. This means that solution to (2) can not be considered even as first approximation to exact solution of our equations. This consideration underlies the reason for singularity at $\mu = 0$.

Algorithm I

Derivation of a singular perturbation series according to A.B.Vasileva and V.F.Butuzov (1973) consist of the following steps. First, take the second term in the exact solution and integrate it by parts to get the following series:

$$\frac{1}{\mu}\int_0^t b(s)\exp\left(-\frac{1}{\mu}\int_t^s a(z)dz\right)ds = -\left[\frac{b(t)}{a(t)} + \mu\left(\frac{b(t)}{a(t)}\right)'\frac{1}{a(t)} + \cdots\right.$$

$$\left. + \left[\frac{b(0)}{a(0)} + \mu\left(\frac{b(0)}{a(0)}\right)'\frac{1}{a(0)} + \cdots\right]\exp\left(\frac{1}{\mu}\int_0^t a(s)ds\right)\right. .$$

Algorithm II

As a result the following series is obtained:

$$x(t) = -\left[\frac{b(t)}{a(t)} + \mu\left(\frac{b(t)}{a(t)}\right)'\frac{1}{a(t)} + \cdots\right] +$$

$$+\left[x_0 + \frac{b(0)}{a(0)} + \mu\left(\frac{b(0)}{a(0)}\right)'\frac{1}{a(0)} + \cdots\right]\exp\left(\frac{1}{\mu}\int_0^t a(s)ds\right).$$

Algorithm III

Let's to make substitution $t = \mu\tau,\ s = \mu\zeta$ in the integrand of the exponent. Then

$$\frac{1}{\mu}\int_0^t a(s)ds = \int_0^\tau a(\mu\zeta)d\zeta = a(0)\tau + \mu\frac{a'(0)}{2}\tau^2 + \mu^2\frac{a''(0)}{6}\tau^3 + \cdots$$

or,

$$\exp\left(\frac{1}{\mu}\int_0^t a(s)ds\right) = e^{a(0)\tau}\left[1 + \mu\frac{a'(0)}{2}\tau^2 + \right.$$

$$\left. + \mu^2\frac{a''(0)}{6}\tau^3 + \mu^2\frac{\tau^4}{4}a'^2(0) + \cdots\right]$$

Algorithm IV

So it has been derived the following series over μ :

$$x(t,\mu) = \tilde{x}(t) + \Pi x(\tau),$$

where

$$\tilde{x}(t) = \tilde{x}_0(t) + \mu\tilde{x}_1(t) + \cdots = -\frac{b(t)}{a(t)} - \mu\left(\frac{b(t)}{a(t)}\right)'\frac{1}{a(t)} + \cdots \quad (3)$$

is a ***regular*** part of the solution and

$$\Pi x(\tau) = \Pi_0 x(\tau) + \mu\Pi_1 x(\tau) + \mu^2\Pi_2 x(\tau) + \cdots \quad (4)$$

is a ***singular*** one with

$$\Pi_0 x(\tau) = \left(x_0 + \frac{b(0)}{a(0)}\right)e^{a(0)\tau};$$

$$\Pi_1 x(\tau) = \left[\left(x_0 + \frac{b(0)}{a(0)}\right)a'(0)\frac{\tau^2}{2} + \left(\frac{b(0)}{a(0)}\right)'\frac{1}{a(0)}\right]e^{a(0)\tau} \quad (5)$$

etc. The terms $\Pi_k x(\tau)$ are called ***boundary layer*** terms.

Algorithm V

Now we can present the algorithm of singular perturbative solution in the following form. Given above mentioned equation with boundary condition for $x(t)$. Then the solution should be divided into two parts as follows: $x(t) = \tilde{x}(t) + \Pi x(\tau)$ and then the equation can be presented in a form:

$$\mu \frac{d\tilde{x}}{dt} + \frac{d\Pi x(\tau)}{d\tau} = a(t)\tilde{x}(t) + a(\mu\tau)\Pi x(\tau) + b(t), \qquad \tau = t/\mu. \quad (6)$$

Further one should to expand each term in both sides of this equation in series over μ. Equating coefficients in front of the same powers of μ, separately for terms depending on t and terms depending on τ, one obtains equations for determination of terms of the expansions (3) and (4).

Back to the QED

Let's to introduce ***new scaled variables*** $\rho = \eta/e$, $\bar{\rho} = \bar{\eta}/e$ and $j_\mu = J_\mu/e$. And then present each field in our main eqs. for QED in the form, divided into regular and singular parts:

$$\psi = \psi^R(J, \eta, \bar{\eta}; e) + \Pi\psi(ej, e\rho,\ e\bar{\rho}; e),$$

$$\overline{\psi} = \overline{\psi}^R(J, \eta, \bar{\eta}; e) + \Pi\overline{\psi}(ej, e\rho,\ e\bar{\rho}; e),$$

$$A_\mu = A_\mu^R(J, \eta, \bar{\eta}; e) + \Pi A_\mu(ej, e\rho,\ e\bar{\rho}; e).$$

Further acting in accordance with above mentioned method one should to divide the QED eqs. into part depending on $J, \eta, \bar{\eta}$ and part, depending on $j, \rho, \bar{\rho}$. Let's for simplification of equations denote the sources as follows: $J, \eta, \bar{\eta} \Leftrightarrow s$ and scaled sources as follows: $j, \rho, \bar{\rho} \Leftrightarrow \sigma$.

Equations for singular parts

Equations for singular (boundary layer) parts are more complicated, I will demonstrate you only one of them:

$$-i\hbar \mathrm{Tr}\left(\frac{\delta \Pi\psi(e\sigma;e)}{\delta\rho}\gamma_\mu\right) = -i\hbar \mathrm{Tr}\left(\frac{\delta \Pi\overline{\psi}(e\sigma;e)}{\delta\overline{\rho}}\gamma_\mu\right) =$$

$$= e\Pi\overline{\psi}(e\sigma;e)\gamma_\mu\psi(e\sigma;e)+$$

$$+e\overline{\psi}(e\sigma;e)\gamma_\mu\Pi\psi(e\sigma;e) + e\Pi\overline{\psi}(e\sigma;e)\gamma_\mu\Pi\psi(e\sigma;e) - D^{-1}_{\mu\nu}\Pi A^\nu(e\sigma;e);$$

Zeroth order equations

At first step we should to extract zeroth order (in e) equations from above mentioned ones:

$$i\hbar \mathrm{Tr}\left(\frac{\delta \Pi_0 \psi(\sigma)}{\delta \rho}\gamma_\mu\right) = i\hbar \mathrm{Tr}\left(\frac{\delta \Pi_0 \overline{\psi}(\sigma)}{\delta \overline{\rho}}\gamma_\mu\right) = D^{-1}_{\mu\nu}\Pi_0 A^\nu(\sigma);$$

$$-i\hbar \frac{\delta \Pi_0 \overline{\psi}(\sigma)}{\delta j_\mu}\gamma_\mu = i\hbar \frac{\delta \Pi_0 \widehat{A}(\sigma)}{\delta \rho} = -\Pi_0 \overline{\psi}(\sigma)\left(i\overleftarrow{\widehat{\partial}} + m\right);$$

$$i\hbar \gamma_\mu \frac{\delta \Pi_0 \psi(\sigma)}{\delta j_\mu} = i\hbar \frac{\delta \Pi_0 \widehat{A}(\sigma)}{\delta \overline{\rho}} = -\left(i\widehat{\partial} - m\right)\Pi_0 \psi(\sigma).$$

Solution to singular part

We have following expression for singular part of the spinor field:

$$\Pi_0\psi(\sigma) = c\left[1 - \exp\left[\frac{i}{4\hbar}\widehat{j}(i\widehat{\partial} - m)\right]\right]\psi_D +$$
$$+ \exp\left[\frac{i}{4\hbar}\widehat{j}(i\widehat{\partial} - m)\right]\sum_{n=0}^{\infty} c_n(\bar{\rho}\rho)^{n+s}\rho =$$
$$= \exp\left[\frac{i}{4e\hbar}\widehat{J}(i\widehat{\partial} - m)\right]\sum_{n=0}^{\infty} c_n(\bar{\rho}\rho)^{n+s}\rho$$

and conjugate expression for $\Pi_0\overline{\psi}(\sigma)$. These expressions has essential singularity at zero coupling limit $e \to 0$ (don't forget $\rho = \eta/e$).

Summary

The singularity at $e = 0$ is very interesting - its existence means that in general we can’t "switch off" electromagnetic interaction and go to "free electron". It is the time to remember Dyson’s proof (1952) that perturbation series in QED is asymptotic one. Our consideration shows that QED effective action can’t be an analytic function in the neighborhood of $e = 0$, consequently, any series in this region can’t be convergent one. In the light of this singularity the notion of "free electron" should be revised - because it is impossible to "switch off" the electromagnetic interaction the existence of free, noninteracting electrically charged particle is questionable. But this point is very hard one and more accurate studies required to be conclusively established.

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