Heavy quark - antiquark potential

Speaker: Rakhimov Nurmukhammad
National University of Uzbekistan

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Inha University
In collaboration with prof.s M. Musakhanov, U. Yakhshiev, H. Ch. Kim

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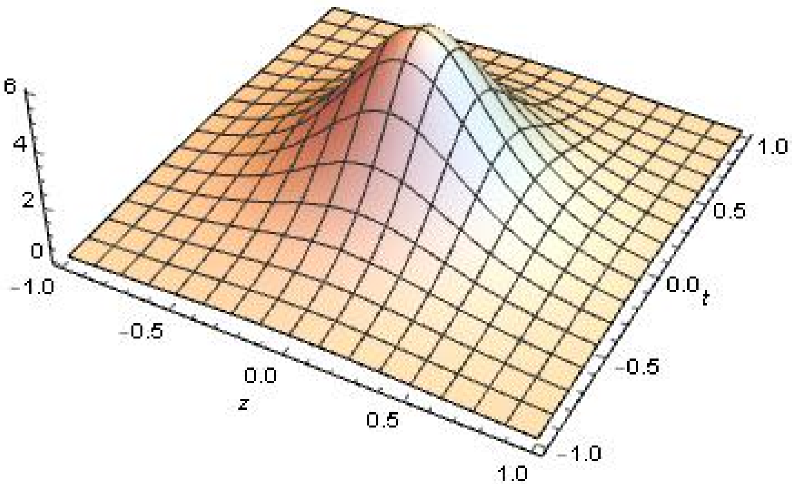
Instantons

- In Euclidean space, a large fluctuation of the gluon field corresponding to quantum tunneling from one vacuum(minimum of the potential energy) to the neighbour one
- The solution of the duality equation

$$F^a_{\mu\nu} = \pm \tilde{F}^a_{\mu\nu}, \qquad A^a_\mu(x) = \frac{2\bar{\eta}^\mu_{\nu a}\rho^2}{x^2(x^2+\rho^2)}$$

- Action density $S = \frac{8\pi^2}{g^2}|Q|$
- $Q = 1$ – instanton,
  $Q = -1$ – anti-instanton



Instanton collective coordinates

- Instantons in $SU(N_c)$ theory can be obtained embedding the $SU(2)$ solution

$$(A_\mu)_{SU(N_c),standard} = \begin{pmatrix} (A_\mu)_{SU(2),standard} & 0 \\ 0 & 0 \end{pmatrix}$$

- $4(centre) + 1(size) + (4N_c - 5)(orientations) = 4N_c$
- Instanton field with arbitrary center z_μ, size ρ and colour orientation U in the $SU(N_c)$ gauge group

$$A_\mu = A_\mu^a t^a = \frac{-i\rho^2 U[\sigma_\mu^-(x-z)^+ - (x-z)_\mu]U^\dagger}{(x-z)^2[\rho^2 + (x-z)^2]},$$

or as

$$A_\mu^a = \frac{2\rho^2 O^{ab}\bar{\eta}^{\mu}_{\nu b}(x-z)_\mu}{(x-z)^2[\rho^2+(x-z)^2]},$$

$$O^{ab} = \mathrm{Tr}(U^\dagger t^a U \sigma^b), \qquad O^{ab}O^{ac} = \delta^{bc}.$$

Instanton Liquid Model

- Instanton liquid characterized by two main parameters:
  Average instanton size – $\rho = \frac{1}{3}\ fm$
  Inter instanton distance – $R = 1\ fm$
- Why liquid?
  - Only a few percent of space occupied by instantons
    $$\rho^4/R^4 \simeq 0.012$$
  - The fields inside the instanton are very strong $G_{\mu\nu} >> \Lambda^2$.
    $$S_0 = 8\pi^2/g^2 \sim 10 - 15 >> 1.$$
  - Instantons retain their individuality and are not destroyed by interactions.
    $$|\delta S_{int}| \sim (2-3) << S_0.$$
  - Nevertheless, interactions are important for the structure of the instanton ensemble, since
    $$\exp|\delta S_{int}| \sim 20 >> 1.$$

The potential from Wilson Loop

- The $Q\bar{Q}$ state at time t is represented as

$$|Q(t,\vec{0})\bar{Q}(t,\vec{r})\rangle = \sum_{C} f(C)\Gamma[(t,\vec{r}),(t,\vec{0});C]|0\rangle$$

Gauge invariant $Q\bar{Q}$ operator

$$\Gamma[x,x';C] = \bar{Q}(x')U(x',x;C)Q(x)$$

$$U(x',x;C) = \exp\left(ig\int_{x}^{x'} \frac{\lambda^a}{2} A_\mu^a(z)dz^\mu\right)$$

- Overlap between the $Q\bar{Q}$ state at $t=0$ and the $Q\bar{Q}$ state at $t=T$ with a complete set of energy eigenstates

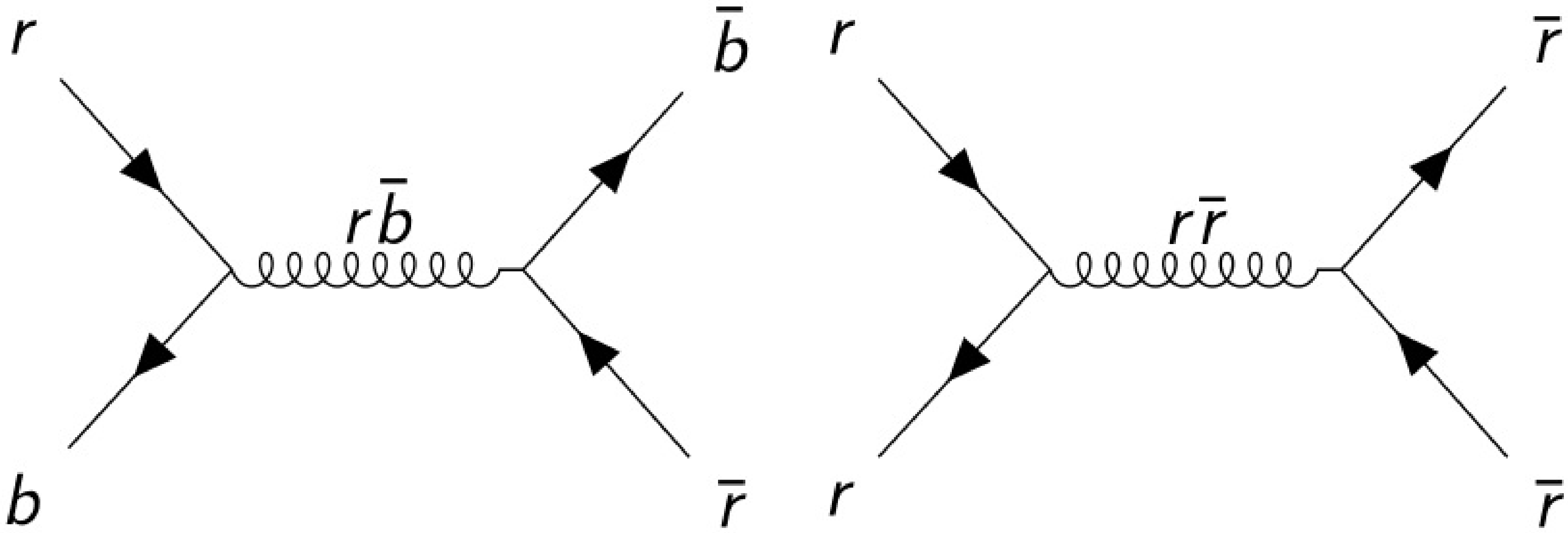
$$\Omega(T,r) = \sum |\langle 0|\Gamma^\dagger[(0,\vec{0}),(0,\vec{r});C]\Gamma[(t,\vec{0}),(t,\vec{r});C]|n\rangle|^2 e^{-E_n T}$$

- For large T, the smallest E_n will dominate and it correspond to the potential energy of the $Q\bar{Q}$ system separated by distance r

$$\lim_{T\to\infty} \Omega(T,r) \sim e^{-V(r)T}$$

Color states

Due to $SU(3)$ there are two kind of color states: octet and singlet



$$\text{Color factor } c = \begin{cases} -\frac{4}{3} \text{ for color singlet state} \\ \\ +\frac{1}{6} \text{ for color octet state} \end{cases}$$

The general picture of the model

- The partition function for the heavy quarks and gluons

$$Z_{\rm full} = Z_Q[\text{quark sources}]Z_g[\text{gluon sources}]$$

- The correlator corresponding to Wilson loop

$$\langle 0|\mathrm{T}\bar{Q}^\dagger Q^\dagger Q\bar{Q}|0\rangle = S_Q S_{\bar{Q}} Z[j]\Big|_{\text{all sources}=0}$$

- The lowest energy state is in Instanton Liquid, thus we need to rewrite the correlator including instantons.

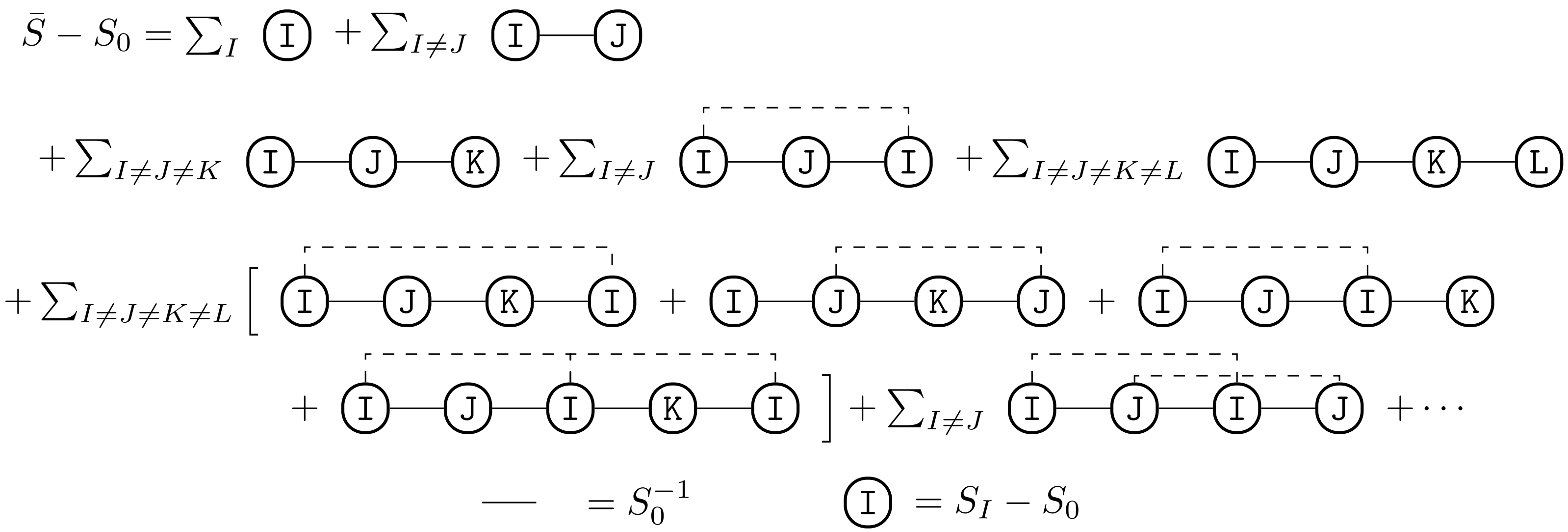
The Pobylitsa Equation

The heavy quark propagator in the instanton media

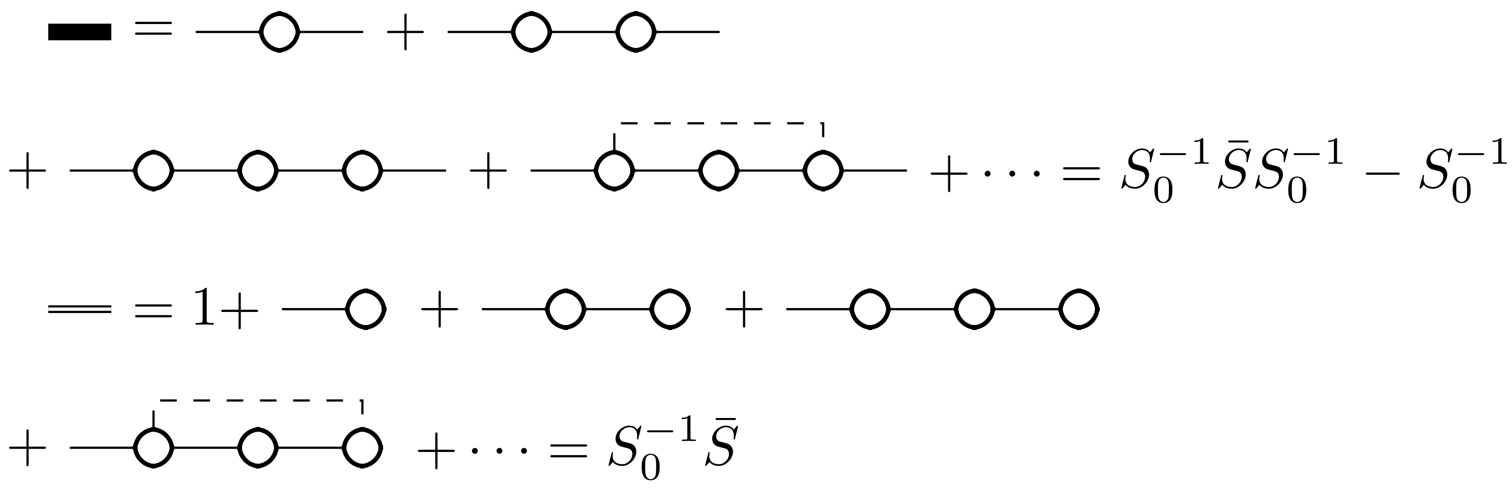
$$\bar{S} = \left\langle \left(S_0^{-1} - \sum_I \hat{A}_I \right)^{-1} \right\rangle$$

Introducing single instanton propagator $S_I = (S_0^{-1} - \hat{A}_I)^{-1}$ the full propagator can be expanded in powers of A_I

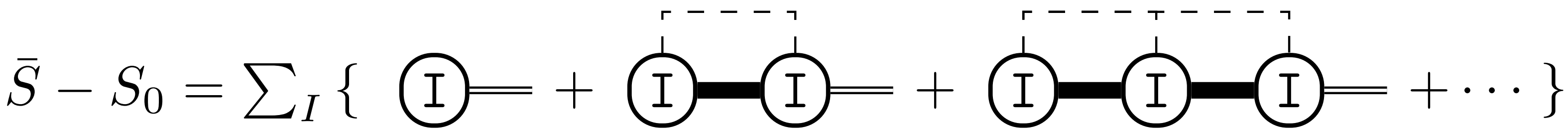
$$\bar{S} - S_0 = \sum_I (S_I - S_0) + \sum_{I \neq J} (S_I - S_0) S_0^{-1} (S_J - S_0) + \ldots$$



- At large N_c the planar graphs dominate
- Average over all other instantons can be summed up



- Skeleton expansion of averaged full propagator



- The averaged propagator for $N/2$ instantons and $N/2$ anti-instantons in 4-dim. volume

$$\bar{S}^{-1} - S_0^{-1} = \frac{N}{2}\left\langle \bar{S} - \hat{A}_I^{-1} \right\rangle + \frac{N}{2}\left\langle \bar{S} - \hat{A}_{\bar{I}}^{-1} \right\rangle$$

- Averaging over z_I – $\langle \cdots \rangle_{z_I} = V^{-1} \int d^4 z_I \cdots$
- Averaging over orientation – $\langle \cdots \rangle_{U_I} = N_c^{-1} Tr_{colour}(\cdots)$
- Rhs $\bar{S}$ can be replaced by S_0, because $\frac{\bar{\rho}^4 N}{VN_c} \simeq 0.004$
- $(S_0 - \hat{A}^{-1})^{-1} = -S_0^{-1}(S_\pm - S_0)S_0^{-1}$, $(+$ for I, $-$ for $\bar{I})$
- Pobylitsa Equation

$$\bar{S}^{-1} = S_0^{-1} - \frac{N}{2VN_c} \sum_{\pm} Tr_c \int d^4 z_I S_0^{-1}(S_\pm - S_0)S_0^{-1} + O((\frac{N}{VN_c})^2).$$

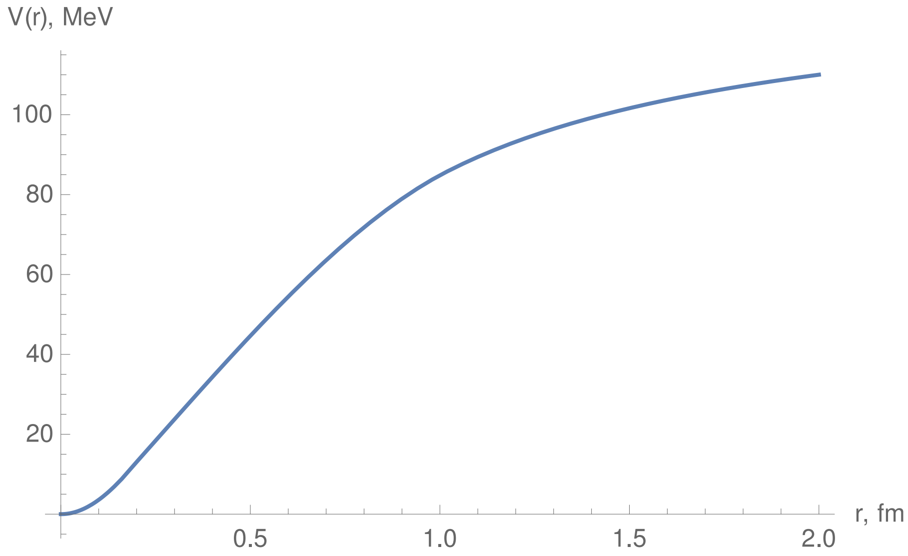
[P. V. Pobylitsa, Phys. Lett. **B**226]

The instanton induced static $Q\bar{Q}$ potential without gluon exchange

- The correlator
  $S_Q S_{\bar{Q}} \sim \left\langle (S_0^{-1} - \sum_I A_I)^{-1} \otimes (S_0^{-1} - \sum_I A_I)^{-1,T} \right\rangle$
- The static potential

$$V(r) = \frac{4\pi\rho^3}{N_C R^4} I\left(\frac{r}{\rho}\right)$$

where $I(x)$ is a dimensionless integral



[U. Yakhshiev et al. ArXiv:1602.06074v5]

Extended form of Pobylitsa Equation

The general field can be decomposed as a sum of instantons ensemble $\sum_I A_{\mu I}$ and a small fluctuations a_μ

$$A_{\rm tot} = \sum_I A_I + a, \qquad DA_{\rm tot} = D\xi Da$$

The action is

$$S(A) = \frac{1}{4}\int d^4x (F^a_{\mu\nu})^2 = \frac{8\pi^2}{g^2} + \int d^4x D_\mu F_{\mu\nu} a_\nu + \int d^4x\, a_\mu W_{\mu\nu} a_\nu + O(a^3)$$

The partition function with the normalization $Z[0] = 1$ and current j of the small fluctuations

$$Z[j] = \int D\xi \exp\left(-\frac{g^2}{2} j_\mu S_{\mu\nu} j_\nu\right), \qquad d\xi_i = V^{-1} d^4 z_i dU_i$$

Extended form of Pobylitsa Equation

- The gluon propagator in lowest on density approximation

$$S_{\rho\nu}(\xi) - S^0_{\rho\nu} = \sum_i (S^i_{\rho\nu}(\xi_i) - S^0{}_{\rho\nu}) + O(N^2)$$

- The partition function changes as

$$Z[j] = \int \exp\left(-\frac{g^2}{2}(jS^0 j)\right) \prod_i \exp\left(-\frac{g^2}{2}(j(S^i - S^0)j)\right) d\xi_i$$

- Averaged heavy quark propagator with gluon fluctuations a

$$w = \left\{ \int \left(\theta^{-1} + \frac{\delta}{\delta j} - \sum_i A_i\right)^{-1} \right.$$
$$\left. \times \exp\left(-\frac{g^2}{2}(jS^0 j)\right) \prod_i \exp\left(-\frac{g^2}{2}(j(S^i - S^0)j)\right) d\xi_i \right\}_{j=0}$$

Extended form of Pobylitsa Equation

$$
\begin{aligned}
w^{-1} = \Big\{ & w_0^{-1}[\frac{\delta}{\delta j}]Z[j] \\
& - \frac{N}{2V}\sum_{\pm}\int d\xi_{\pm} w_0^{-1}[\frac{\delta}{\delta j}]\left(w_{\pm} - w_0[\frac{\delta}{\delta j}]\right) \\
& \times w_0^{-1}[\frac{\delta}{\delta j}]\exp(-\frac{g^2}{2}(jS^{\pm}j))\Big\}_{j=0}
\end{aligned}
$$

where

- $w_0[\frac{\delta}{\delta j}] = \left(\theta^{-1} + \frac{\delta}{\delta j}\right)^{-1}$ – "free" propagator
- $w_{\pm}[\frac{\delta}{\delta j}] = \left(\theta^{-1} + \frac{\delta}{\delta j} - A_{\pm}\right)^{-1}$ – single (anti-)instanton($\pm$) propagator

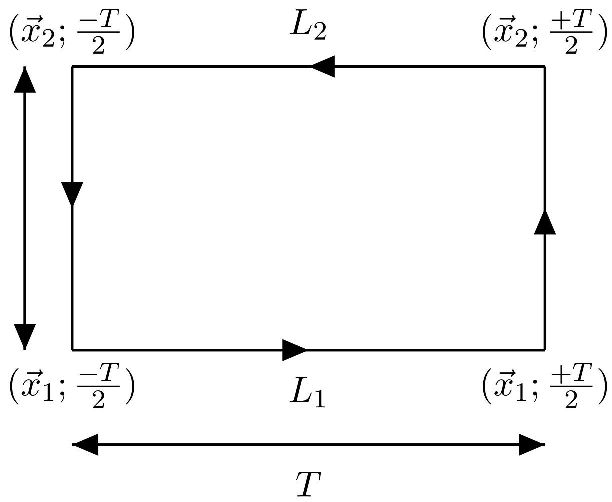
One gluon exchange potential

- Pobylitsa Eq. for 4-point correlator in leading order to instantons density N/V

$$W_0^{-1} = \left\{ \left(w_0^{(1)}[\frac{\delta}{\delta j^{(1)}}]^{-1} \otimes w_0^{(2)}[\frac{\delta}{\delta j^{(2)}}]^{-1,T} \right) Z[j] \right\}_{j=0}$$
$$= \theta^{-1} \otimes \theta^{-1} + g^2 \int D\xi S_{44}(\xi, r)$$

- One gluon exchange potential of the $Q\bar{Q}$ system

$$V_{1gl} = cg^2 \bar{S}_{44}(r)$$



The momentum dependent effective gluon mass

- The gluon propagator averaged over instantons collective coordinates

$$\bar{S}_{\rho\nu} = S^0_{\rho\nu} + N(\bar{S}^I_{\rho\nu} - S^0_{\rho\nu}) \sim \delta_{\mu\nu}/(q^2 + M_g^2(q))$$

- Effective gluon mass

$$M_g(q) = \left(\frac{24\pi^2\rho^2}{(N_C^2-1)R^4}\right)^{1/2} q\rho \mathrm{K}_1(q\rho)$$

Figure of Mass

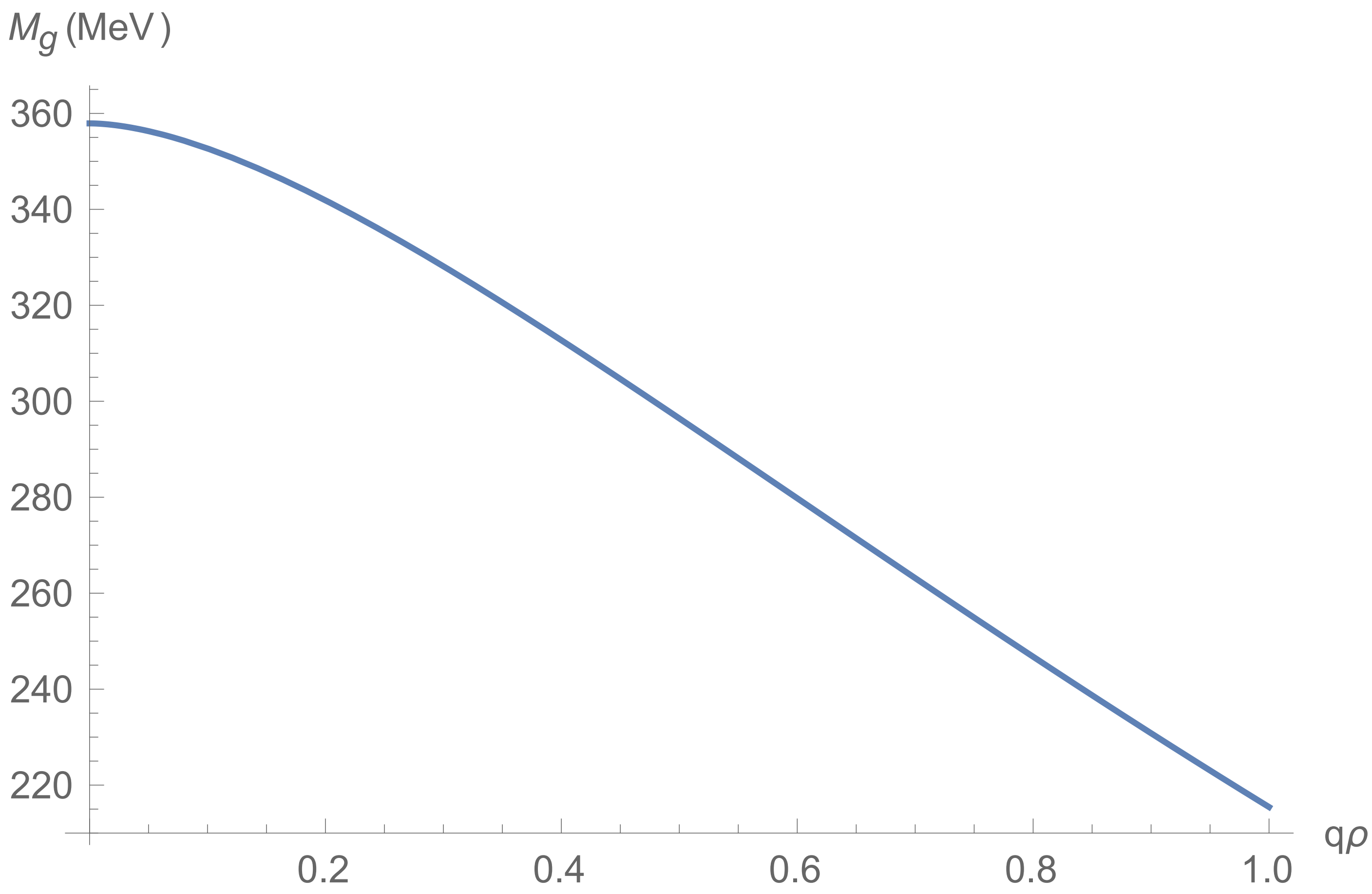


Figure: Dynamical effective gluon mass

The one-gluon exchange potential in coordinate representation

$$V(r) = -\frac{4\alpha_s}{3}\left\{\frac{1}{r} - \frac{2}{\pi\rho} I\left(\frac{r}{\rho}\right)\right\}$$

where $I(x)$ – dimensionless integral with asymtotics

$$\lim_{x\to 0} I(x) = \mathrm{const}, \qquad \lim_{x\to\infty} = \frac{\pi}{2x}$$

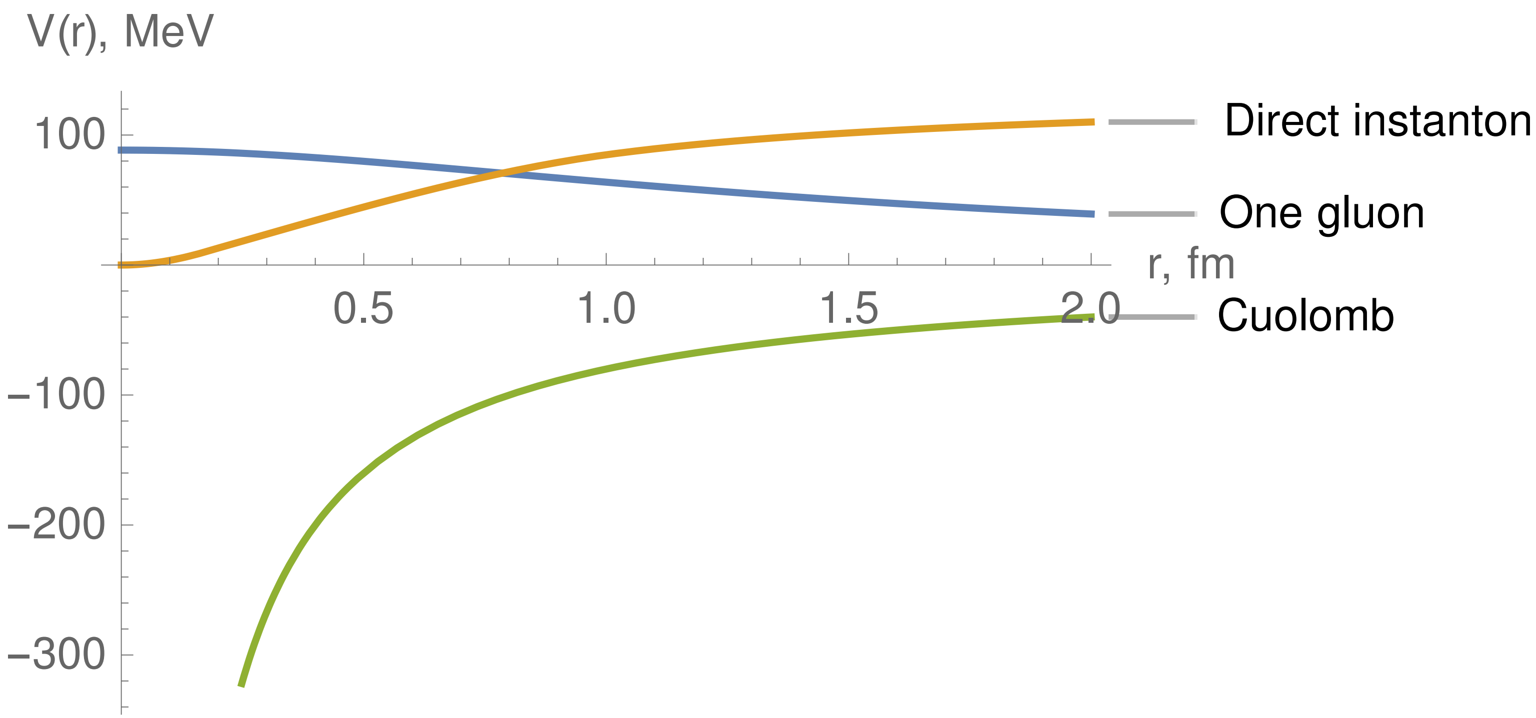
The asymtotics of the potential

$$V(r\to 0) \longrightarrow -\frac{4\alpha_s}{3r} + \mathrm{const}$$
$$V(r\to\infty) \longrightarrow 0$$

The phenomenology of heavy quarks

The $Q\bar{Q}$ potential ingredients

$$V_{Q\bar{Q}}(r) = V_{\rm Conf.}(r) + V_{\rm Cuolomb}(r) + V_{\rm dir.ins.}(r) + V_{\rm 1gl}(r) + V_{\rm SD}(r)$$



How to find spin dependent parts of the potential

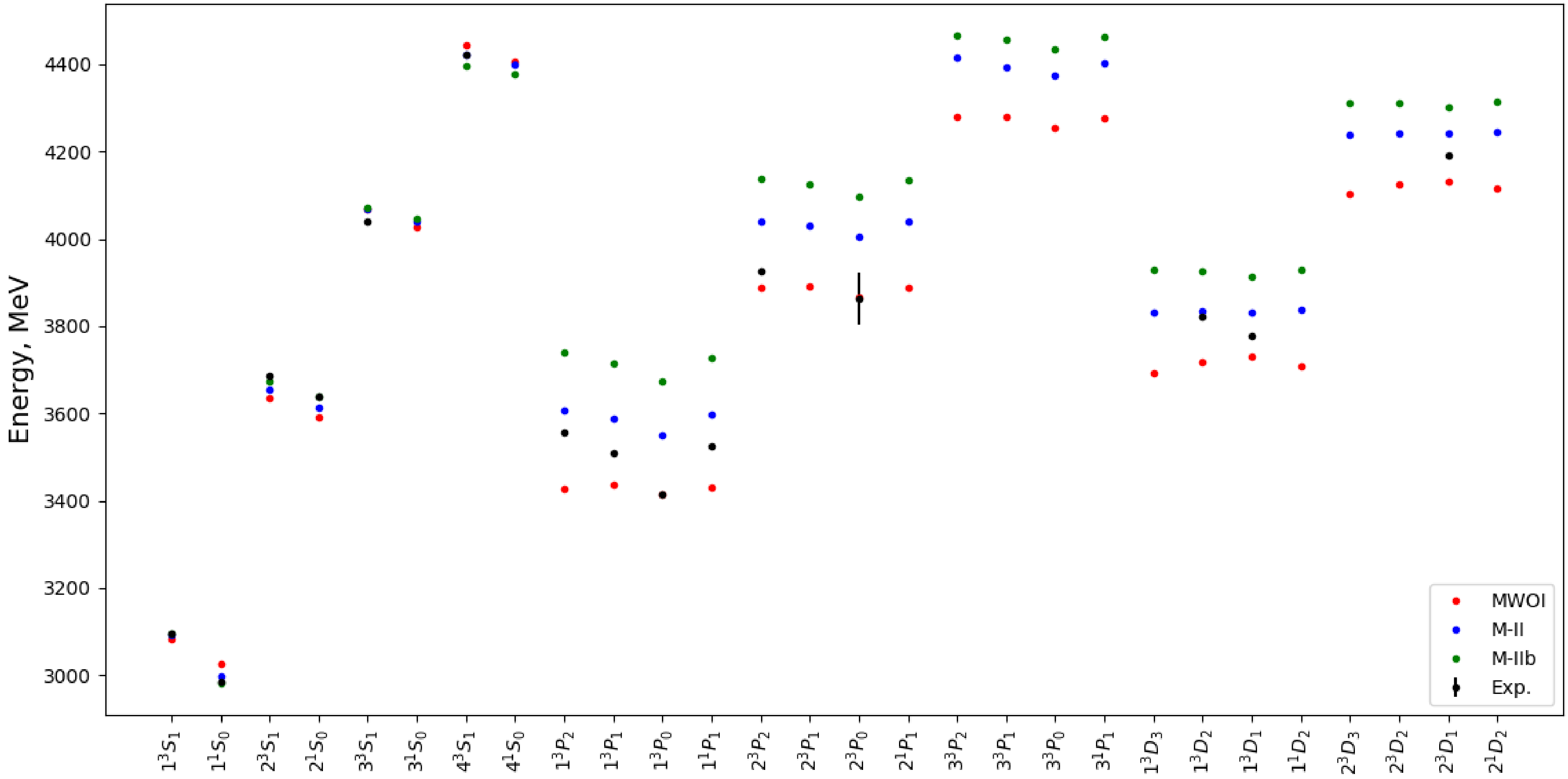
- Make Foldy-Wouthuysen transformation – series expansion over m_Q^{-1}
- The potential can be described as

$$V_{full} = V_C + c_1 V_V + c_2 V_{LS} + c_3 V_{SS} + c_4 V_T$$

- Also can be obtained by Breit Equation

$$V_{SD}(r) = (\vec{L}\vec{S}) \overbrace{\frac{3}{2m^2}\frac{1}{r}\frac{dV_c(r)}{dr}}^{V_{LS}(r)} + \left(\vec{S}^2 - \frac{3}{2}\right) \overbrace{\frac{1}{3m^2}\Delta V_c(r)}^{V_{SS}(r)}$$
$$+ \left(\vec{S}^2 - \frac{3(\vec{S}\vec{r})(\vec{S}\vec{r})}{r^2}\right) \underbrace{\frac{1}{6m^2}\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right] V_c(r)}_{V_T(r)}$$

Comparison of previous results



Thanks for attention!!!